Research Article

Flexure Analysis of Functionally Graded Plates Using \{2,2\}-Refined Zigzag Theory

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Abstract

This study investigates the flexure analysis of functionally graded (FG) plates using \{2,2\}-refined zigzag plate theory which considers transverse normal deformation along the thickness of plates. This element eliminates the use of the shear correction factors. The FG plate is composed of silicon carbide (SiC) and aluminum (Al) varying through the thickness of the plate. The volume fractions of the material constituents in the FG plate were functionally tailored based on a power-law. The effective material properties of the plate were evaluated by using the Mori-Tanaka homogenization method. The accuracy of the present approach was demonstrated by considering a simply supported FG plate under distributed sinusoidal load. The influence of the through-thickness material variation on the stress and displacement distributions was investigated. It was observed that the material variation through the thickness played a major role on the stress and displacement levels whilst the influence of the material variation was minor on the stress and displacement profiles.


Geliştirilmiş Zigzag Teorisi Kullanarak Fonksiyonel Kademelendirmiş Plakların Eğilme Analizleri

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Öz

1. INTRODUCTION

Classical multi-layered composites are comprised of distinct materials and have been widely used in many engineering fields such as nuclear, military, automotive, and aerospace for many years. However, they suffer from the discontinuity of the thermal and mechanical properties between the layers of composites. Therefore, the interfacial stress concentrations may take place leading to crack nucleation and debonding between the layers of the composite structures. The functionally graded materials (FGMs) were proposed in the 1980s to eliminate the aforementioned interfacial problems [1,2]. A typical FGM is a heterogeneous material consists of two distinct constituents, such as metal and ceramic, with a volume fraction varying continuously. This continuous variation can be obtained for mechanical and thermal properties which help to minimize the stress concentrations at interfaces between two different materials. As FGMs are vital of importance in critical engineering fields, the deformation and stress behaviors under loads should be well understood for their structural integrity.

Finite Element Analysis (FEA) has been commonly used to examine the strength of FGMs. In order to assess the strength of the FGMs, there is a necessity to obtain accurate strain and stress fields. Hence, modeling of such structures may be impractical by using the traditional 3-D finite elements due to the extremely mesh refinement. There exists extensive effort on the development of layerwise plate theories. However, their performance on the calculation of the stress and strain fields may not be satisfactory due to the transverse shear correction factors and the representation of the stress fields [3]. Alternatively, Tessler et al. [3-5] developed a refined zigzag theory (RZT) by eliminating the requirement for the shear correction factors. Also, RZT-based finite element formulations were developed for the analysis of beam and plate structures [6-8]. Madenci et al. [9,10] developed a new $C^0$ continuous {2,2}-refined zigzag plate theory (RZT) which captures the transverse normal deformation (stretch). The proposed RZT is introduced by the notation {2,2} where the first number indicates the order of expansion used for the in-plane displacement component while the second number denotes the order of expansion for the transverse displacement component. Moreover, they examined the behavior of the laminates and sandwich constructions with and without internal stiffeners under static and dynamic loads including the effect of the change in temperature [11-15].

Numerous research has been recently presented to provide an in-depth understanding of the behavior of FGMs under dynamic and static loads [16-19]. Swaminathan et al. [20] presented a detailed review on the methods used to investigate the static and dynamic behavior of FG plates. They considered both analytical and numerical methods. They investigated the influence of variation of material properties, boundary and loading conditions, and nonlinearity on the behavior of FG plates. Reddy [21] proposed both analytical and finite element models based on higher order shear deformation theories (HSDT) for static and dynamic analyses of FG plates. Zenkour [22] developed an analytical technique to examine the response of the FG plate under a transverse uniform load using the generalized shear deformation theory (GSDT). A stress analysis was carried out for an isotropic FG plate assuming a power law distribution in gradient. The results were compared with those in an equivalent homogeneous isotropic plate. Nguyen et al. [23] calculated shear correction factors to examine the static analysis of rectangular sandwich and FG plates by using first-order shear deformation (FSDT). They showed that the shear correction factor depended on the material distribution and ratio between elastic moduli of constituents. They predicted the transverse shear stresses by integrating the 3D elasticity equations. They revealed that the transverse shear stresses were in good agreement with those of third-order shear deformation theory (TSDT). Ferreira et al. [24] investigated the static deformations of FG plates using meshless collocation method and the TSDT. They examined the influence of the aspect ratio of the plate and the compositional gradient component of the constituents on the centroidal deflection. The results obtained using two different homogenization schemes were identical to each other when the Poisson’s ratios of the two material constituents were nearly equal. However, the results from the two homogenization schemes were quite different for widely varying Poisson’s ratios of the two constituents. Iurlaro et al. [25] studied bending and free vibration analysis of FG sandwich plates using the RZT. It was assumed that transverse displacement was constant across thickness; thus, their kinematic representations did not take into account transverse stretching. Also, they indicated that RZT can be efficiently used for both traditional and FG sandwich structures.

This study aims to investigate the flexural analysis of FG plates using {2,2}-refined zigzag plate theory recently developed by Barut et al. [9,10]. The element employs anisoparametric shape functions so that it does not suffer from geometric locking. The anisoparametric interpolations consider shape functions for the transverse displacements that are one degree higher than those used for the bending rotations [8]. Also, the element is free of shear correction factor because each layer has a constant shear strain variation. The zigzag functions have individual layer shear rigidity, and the slopes of the zigzag functions vanish when summed through the thickness [3-5]. In the analyses, the FG plate has a...
composition variation of SiC and Al powders through the thickness. The through-thickness local mechanical properties of the FG plate were evaluated by using the Mori-Tanaka homogenization method [26]. The governing equations of the FG plate and the boundary conditions were derived by employing the principle of virtual work. Static results were presented for the simply supported FG plates under the transverse sinusoidal load. The influence of the material variation on the displacement and stress fields was investigated.

2. EFFECTIVE MATERIAL PROPERTIES

In this study, the FG plate is comprised of a homogeneous mixture of metal and ceramic phases. The top layer of the FG plate is pure ceramic and its bottom layer is pure metal. The through-thickness mechanical properties of the FG plate are evaluated by using the Mori-Tanaka homogenization scheme [26]. The volume fractions of two constituents are of the form

\[ V_1 + V_2 = 1 \]  

where the subscripts 1 and 2 refer to the metal (Al) and ceramic (SiC) phases, and \( V_1(z) \) and \( V_2(z) \) denote the volume fractions of the metal and ceramic phases, respectively. The volume fraction of the metal and ceramic phases can be expressed as

\[ V_1(z) = \left(1 - \frac{2z + h}{2h}\right)^n, \]  

and

\[ V_2(z) = 1 - V_1(z) \]

respectively. The thickness of the plate is \( 2h \), and \( n \) represents the compositional gradient exponent varies from 0.1 (metal-rich) to 10 (ceramic-rich). Figure 1 illustrates the through-thickness variations of the volume fraction of the metal phase for the compositional gradient exponents of \( n = 0.1, 0.5, 1, 2 \) and 10 in the FG plate.

The bulk \( K \) modulus of the FG plate can be expressed as

\[ K(z) = K_1 + \frac{V_2(K_3 - K_1)}{1 + (1 - V_2) \frac{3(K_2 - K_1)}{3K_1 + 4G_1}} \]  

and the shear modulus

\[ G(z) = G_1 + \frac{V_2(G_3 - G_1)}{1 + (1 - V_2) \frac{(G_2 - G_1)}{G_1 + f_1}} \]

in which

\[ f_1 = \frac{G_i(9K_2 + 8G_2)}{6(K_2 + 2G_2)} \]  

Young’s modulus is

\[ E(z) = \frac{9KG}{3K + G} \]  

and Poisson’s ratio

\[ \nu(z) = \frac{3K - 2G}{2(3K + G)} \]

The density \( \rho \) is calculated by considering the linear rule of mixtures as

\[ \rho(z) = V_1 \rho_1 + V_2 \rho_2 \]

Figure 1. Through thickness variations of the volume fraction of the metal constituent.

3. ELEMENT DESCRIPTION

Figure 2 shows the description of six-node triangular refined zigzag element (RZ122) with its nodal degrees of freedom (DOF). As seen, there are three nodes at the corners and mid-side nodes along the edges. The corner nodes have eleven degrees of freedom (DOF) while the mid-side nodes include three DOF resulting in 42 DOF. Also, the corner nodes have two in-plane displacement components, \( u^{(i)} \) and \( v^{(i)} \); transverse displacement components, \( w_1^{(i)} \) and \( w_2^{(i)} \); average rotations, \( \theta_1^{(i)} \) and \( \theta_2^{(i)} \); and first and second modes of out-of-plane...
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zigzag rotations, \( \psi_1^{(1)}, \psi_2^{(1)}, \hat{\psi}_1^{(1)}, \) and \( \hat{\psi}_2^{(1)} \) with \( i = 1, 2, 3 \). The mid-side nodes only consist of the transverse displacement components, \( w_1^{(i+3)} \), \( w_2^{(i+3)} \), and \( w_1^{(i+3)} \) \( (i = 1, 2, 3) \). These additional transverse-displacement variables, \( w_1^{(i+3)} \) and \( w_2^{(i+3)} \), represent the symmetric and anti-symmetric thickness-stretch modes.

\[ u_i(x_1, x_2, z) = u(x_1, x_2) + \]
\[ z \theta_i(x_1, x_2) + \phi_i(z) \psi_i(x_1, x_2) + \] \[ \frac{z}{h} \phi_i(z) \hat{\psi}_i(x_1, x_2) \]
\[ v_i(x_1, x_2, z) = v(x_1, x_2) + \]
\[ z \theta_i(x_1, x_2) + \phi_i(z) \psi_i(x_1, x_2) + \] \[ \frac{z}{h} \phi_i(z) \hat{\psi}_i(x_1, x_2) \]
\[ w_i(x_1, x_2, z) = w(x_1, x_2) + \] \[ \frac{z}{h} \psi_i(x_1, x_2) \]

in which \( z \) is the thickness coordinate. As defined by Tessler [27], an average transverse normal stress, \( \sigma_{zz} \), is assumed as a cubic function through the plate thickness as

\[ \sigma_{zz}(x, z) = \sigma_{zz}(x) + \sigma_{zz}(x) \left( \frac{z}{h} - \frac{z^3}{3h^3} \right) \]

in which \( \sigma_{zz}(x) \) and \( \sigma_{zz}(x) \) are functions of the kinematic variables in Eq. (10). For further details on the derivation of Eqs. (10), (11), the readers should refer to [9,10].

The piecewise-linear \( C^0 \)-continuous functions through the thickness of the plates are employed for the zigzag kinematics which describe cross-sectional distortions. Thus, the use of shear correction factor can be eliminated when modeling the homogeneous and heterogeneous laminates.

4. KINEMATICS OF \( \{2,2\}\)-RZE

The Cartesian displacements in the \( k \)-th layer in a plate are expressed in terms of the average in-plane displacements \((u, v)\), average transverse displacements \((w, w_1 \) and \( w_2)\), average slopes \((\theta_1 \) and \( \theta_2)\), and the zigzag amplitudes, \((\psi_\alpha \) and \( \hat{\psi}_\alpha \) with \( \alpha = 1, 2 \)) as

\[ u_i(x_1, x_2, z) = u(x_1, x_2) + \]
\[ z \theta_i(x_1, x_2) + \phi_i(z) \psi_i(x_1, x_2) + \] \[ \frac{z}{h} \phi_i(z) \hat{\psi}_i(x_1, x_2) \]
\[ v_i(x_1, x_2, z) = v(x_1, x_2) + \]
\[ z \theta_i(x_1, x_2) + \phi_i(z) \psi_i(x_1, x_2) + \] \[ \frac{z}{h} \phi_i(z) \hat{\psi}_i(x_1, x_2) \]
\[ w_i(x_1, x_2, z) = w(x_1, x_2) + \] \[ \frac{z}{h} \psi_i(x_1, x_2) \]

5. ELEMENT DEVELOPMENT

As described in Fig. 2, the kinematic fields of the \( \{2,2\}\)-RZE can be approximated using linear shape functions for the in-plane displacements, bending rotations, and zigzag amplitudes, and a quadratic shape function for the transverse displacement components, the element interpolations in terms of the linear area-parametric coordinates

\[ u_i(x_1, x_2, z) = u(x_1, x_2) + \]
\[ z \theta_i(x_1, x_2) + \phi_i(z) \psi_i(x_1, x_2) + \] \[ \frac{z}{h} \phi_i(z) \hat{\psi}_i(x_1, x_2) \]
\[ v_i(x_1, x_2, z) = v(x_1, x_2) + \]
\[ z \theta_i(x_1, x_2) + \phi_i(z) \psi_i(x_1, x_2) + \] \[ \frac{z}{h} \phi_i(z) \hat{\psi}_i(x_1, x_2) \]
\[ w_i(x_1, x_2, z) = w(x_1, x_2) + \] \[ \frac{z}{h} \psi_i(x_1, x_2) \]


\[ u(x_i, x_j) = \sum_{i=1}^{3} \zeta_i u^{(i)} \]

\[ v(x_i, x_j) = \sum_{i=1}^{3} \zeta_i v^{(i)} \]

\[ w(x_i, x_j) = \sum_{i=1}^{3} L_i w^{(i)} \]

\[ \{ w_1 (x_i, x_j), w_2 (x_i, x_j) \} = \sum_{i=1}^{3} L_i \{ w_1^{(i)}, w_2^{(i)} \} \] (12)

\[ \theta_{\alpha} (x_i, x_j) = \sum_{i=1}^{3} \zeta_i \theta_{\alpha}^{(i)} \]

\[ \dot{\psi}_{\alpha} (x_i, x_j) = \sum_{i=1}^{3} \zeta_i \dot{\psi}_{\alpha}^{(i)} \]

\[ \ddot{\psi}_{\alpha} (x_i, x_j) = \sum_{i=1}^{3} \zeta_i \ddot{\psi}_{\alpha}^{(i)} \] with \( (\alpha = 1, 2) \)

where \( \zeta_i (i = 1, 2, 3) \) and \( L_i (i = 1, \ldots, 6) \) are area coordinates and quadratic shape functions, respectively. The area coordinates can be expressed as

\[ \zeta_i (x_i, x_j) = \frac{1}{2A} (c_i + b_i x_i + a_i x_j) \] (13)

with

\[ c_i = x_i^{(j)} x_j^{(k)} - x_i^{(j)} x_j^{(k)} \]

\[ b_i = x_j^{(j)} - x_j^{(k)} \]

\[ a_i = x_j^{(k)} - x_i^{(j)} \] (14)

where \( i = 1, 2, 3, j = 2, 3, 1, \) and \( k = 3, 1, 2 . \) The quadratic shape functions, \( L_i \), can be expressed in terms of area coordinates \( \zeta_i \) as

\[ L_i (x_i, x_j) = 2\zeta_i \left( \zeta_i - \frac{1}{2} \right) \]

\[ L_{i+j} (x_i, x_j) = 4\zeta_i \zeta_j \] (15)

with \( i = 1, 2, 3 \) and \( j = 2, 3, 1 . \)

Eq. (12) can be written in matrix form as

\[ \mathbf{u}_x = \mathbf{N}_x \mathbf{v}_x \] (16)

where

\[ \mathbf{u}_x = \{ u^{(k)}, v^{(k)}, w^{(k)}, w_1^{(k)} \} \]

\[ \theta_1^{(k)}, \theta_2^{(k)}, \psi_1^{(k)}, \psi_2^{(k)}, \dot{\psi}_1^{(k)}, \dot{\psi}_2^{(k)} \} \] (17)

\[ \mathbf{v}_x = \{ u^{(i)}, v^{(i)}, w^{(i)}, w_1^{(i)} , \theta_1^{(i)}, \theta_2^{(i)}, \psi_1^{(i)}, \psi_2^{(i)} , \dot{\psi}_1^{(i)}, \dot{\psi}_2^{(i)} \} \]

\[ u^{(2)}, v^{(2)}, w^{(2)}, w_1^{(2)} \]

\[ u^{(3)}, v^{(3)}, w^{(3)}, w_1^{(3)} \]

\[ w^{(4)}, w_1^{(4)}, w_2^{(4)}, w_1^{(5)}, w_2^{(5)}, w^{(6)}, w_1^{(6)}, w_2^{(6)} \} \]

\[ \{ u^{(2)} , v^{(2)} , w^{(2)} , w_1^{(2)} , \theta_1^{(2)}, \theta_2^{(2)}, \psi_1^{(2)}, \psi_2^{(2)}, \dot{\psi}_1^{(2)}, \dot{\psi}_2^{(2)} \} \]

\[ \{ u^{(3)} , v^{(3)} , w^{(3)} , w_1^{(3)} , \theta_1^{(3)}, \theta_2^{(3)}, \psi_1^{(3)}, \psi_2^{(3)}, \dot{\psi}_1^{(3)}, \dot{\psi}_2^{(3)} \} \]

\[ \{ u^{(4)} , v^{(4)} , w^{(4)} , w_1^{(4)}, w_2^{(4)}, w_1^{(5)}, w_2^{(5)}, w^{(6)}, w_1^{(6)}, w_2^{(6)} \} \] (18)

and

\[ \mathbf{N}_x = \begin{bmatrix} \mathbf{N}_u & \mathbf{N}_v & \mathbf{N}_w & \mathbf{N}_{w_1} & \mathbf{N}_{w_2} & \mathbf{N}_{\theta_1} & \mathbf{N}_{\theta_2} & \mathbf{N}_{\psi_1} & \mathbf{N}_{\psi_2} & \mathbf{N}_{\psi_3} \end{bmatrix} \] (19)

where the shape function vectors \( \mathbf{N} \) are described in detail in Ref. [10].

Based on RZE formulation, the strain resultant vector can be written as

\[ \mathbf{e}_x = \mathbf{B}_x \mathbf{v}_x \] (20)

Substituting for the element interpolations (Eq. (12)) in the strain resultant vector (Eq. (20)) yields

\[ \mathbf{e}_x = \mathbf{B}_x \mathbf{v}_x \] (21)

where \( \mathbf{B}_x \) denotes the resultant strain-displacement transformation matrix [10].

6. RESULTANT STRESS- STRAIN RELATIONS

The relationship between the resultant stress and resultant strain in the global coordinate system can be expressed in the form

\[ \mathbf{s}_x = \mathbf{H} \mathbf{e}_x \] (22)

where \( \mathbf{H} \) represents the material property matrix of the plate [9, 10]. In Eq. (22), the resultant stress vector \( \mathbf{s}_x \) contains

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\( s_{e} = \begin{bmatrix} N_{11}, N_{22}, N_{33}, N_{12}, \\ M_{12}, M_{22}, M_{33}, M_{12}, \\ \hat{M}_{11}, \hat{M}_{22}, \hat{M}_{33}, \hat{M}_{21}, \\ Q_{1}, Q_{2}, \hat{Q}_{1}, \hat{Q}_{2}, \\ \hat{\hat{Q}}_{1}, \hat{\hat{Q}}_{2}, \hat{\hat{Q}}_{1}, \hat{\hat{Q}}_{2}, \\ \hat{\hat{\hat{Q}}}_{1}, \hat{\hat{\hat{Q}}}_{2}, \hat{\hat{\hat{Q}}}_{1}, \hat{\hat{\hat{Q}}}_{2} \end{bmatrix} \) \hspace{1cm} (23)

where, \( N_{ab} \) refer to in-plane stress resultants, \( (M_{ab}, \hat{M}_{ab}) \) represent bending stress resultants, and \( (Q_{ab}, \hat{Q}_{ab}, \hat{\hat{Q}}_{ab}, \hat{\hat{\hat{Q}}}_{ab}) \) with \( (\alpha, \beta=1,2) \) denote transverse shear stress resultants of the plate.

7. EQUILIBRIUM EQUATIONS

The element equilibrium equations of the plate element is derived based on the principal of virtual work given by

\[ \partial W_{f} + \partial W_{E} = 0 \] \hspace{1cm} (24)

where \( \partial W_{f} \) is the virtual work done by the internal forces and \( \partial W_{E} \) refers to the work done by the external forces. The internal virtual work, \( \partial W_{f} \), are of the form

\[ \partial W_{f} = \int_{A_{e}} \partial \varepsilon_{e}^T s_{e} dA \] \hspace{1cm} (25)

Substituting for the resultant strain and stress vectors Eqs. (21) and (22) into Eq. (25) leads to the compact matrix form of the internal virtual work as

\[ \partial W_{f} = \delta \varepsilon_{e}^T k_{e} \varepsilon_{e} \] \hspace{1cm} (26)

in which \( k_{e} \) is the element stiffness matrix which can be expressed as

\[ k_{e} = \int_{A_{e}} B_{e}^T H B_{e} dA \] \hspace{1cm} (27)

The virtual work due to external forces can be expressed as

\[ \partial W_{E} = \delta \varepsilon_{e}^T f_{e} \] \hspace{1cm} (28)

in which \( f_{e} \) denotes the external force vector.

Substituting for Eqs. (26) and (28) into Eq. (24) leads to the element equilibrium equations

\[ k_{e} \varepsilon_{e} = f_{e} \] \hspace{1cm} (29)

The global equilibrium equation can be expressed as

\[ K \varepsilon = F \] \hspace{1cm} (30)

where \( K \) is the global stiffness matrix, \( \varepsilon \) is the structural nodal unknown vector, and \( F \) represents external load vector.

8. NUMERICAL RESULTS

A FG plate is composed of SiC and Al along the thickness between the top ceramic (SiC) surface and the bottom metal (Al) surface as shown in Fig. 4. The simply supported FG plate is under a sinusoidal load specified in the form

\[ q = q_{0} \sin \left( \frac{\pi x_{1}}{L} \right) \cos \left( \frac{\pi x_{2}}{W} \right) \] \hspace{1cm} (31)

where \( q_{0} = -1 \) N/m\(^2\) indicates the magnitude of the sinusoidal load. Also, the width and length of the plate are specified as \( W = L = 0.2 \) m. The thickness of the FG plate is \( 2h = 0.02 \) m which is moderately thick.

The simply supported boundary conditions all four edges are given in the form of \( u_{x} = u_{x} = 0 \) along \( x_{1} = 0, L \) and \( u_{y} = u_{y} = 0 \) along \( x_{2} = 0, W \).
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Figure 4. Simply supported FG plate under sinusoidal loading.

These point-wise boundary conditions can be expressed in terms of the displacement fields as

\[ v = w = w_1 = w_2 = \theta_1 = \psi_1' = \psi_2' = 0 \]

along \( x_1 = 0, L \) \hspace{1cm} (32)

and

\[ u = w = w_1 = w_2 = \theta_1' = \psi_1 = \psi_1' = 0 \]

along \( x_2 = 0, W \) \hspace{1cm} (33)

The isotropic material properties of Al and SiC constituents are specified as Young’s modulus of 70 GPa and 300 GPa and Poisson’s ratio of 0.3 and 0.17, respectively. The influence of the through-thickness material variation of the FG plate on the displacement and stress fields was investigated for the compositional gradients of \( n = 0.1 \) (metal-rich), 0.5, 1, 2, and 10 (ceramic-rich).

In order to predict an accurate solution, a convergence study is performed by considering the refinement of the meshes as shown in Fig 5. Also, the validity of the present approach is established and demonstrated with the analytic solution technique proposed by Anderson et al. [28].

![Refinement of the finite element meshes](image)

Figure 5. Refinement of the finite element meshes for the convergence study: (a) \( n' = 1 \), (b) \( n' = 3 \), and (c) \( n' = 5 \).

Figure 6 depicts the convergence plots for the transverse displacement at the center of the FG plate for the linear composition variation \(( n = 1 \)). As seen, increasing the number of elements in the solution domain provides more accurate results. Based on the convergence study, a relatively fine uniform mesh of \( n' = 25 \) is used for the analyses. Therefore, the solution domain is constructed with 2500 uniformly distributed triangular elements resulting in 11097 DOF.

![Convergence of transverse displacement](image)

Figure 6. Convergence of transverse displacement at the center of the FG plate for \( n = 1 \). The through-thickness variations of the displacement and stress components are determined at three evaluation points as shown in Fig. 7. Tables 1 and 2 compare the mid-plane \(( z = 0 \) m\) displacements and transverse stresses for the compositional gradient exponents of \( n = 0.1, 0.5, 1, 2, \) and 10 at the evaluation points (Fig. 7). The displacement and stress results for each material variation agree well with the analytical solutions. Figure 8 shows the comparison of the through-thickness in-plane and transverse displacement components at the evaluation points (Fig. 7) for the linear material composition \( n = 1 \). It is obvious that the through-thickness variations of the in-plane displacements are successfully predicted by the \( \text{RZE}^{(2,2)} \).
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Figure 7. Evaluation points of displacement and stress components.

Table 1. Comparisons of the in-plane and transverse displacements at mid-plane for \( n = 0.1, 0.5, 1, 2, \) and 10.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( u \times 10^{-12} ) (at point ( B ))</th>
<th>( v \times 10^{-12} ) (at point ( C ))</th>
<th>( w \times 10^{-11} ) (at point ( A ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical</td>
<td>RZE (^{2,2})</td>
<td>Analytical</td>
<td>RZE (^{2,2})</td>
</tr>
<tr>
<td>0.1</td>
<td>0.7213</td>
<td>0.7211</td>
<td>0.9422</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0892</td>
<td>1.0889</td>
<td>1.4230</td>
</tr>
<tr>
<td>1</td>
<td>1.0933</td>
<td>1.0933</td>
<td>1.4285</td>
</tr>
<tr>
<td>2</td>
<td>0.8940</td>
<td>0.8934</td>
<td>1.1702</td>
</tr>
<tr>
<td>10</td>
<td>0.2826</td>
<td>0.2818</td>
<td>0.3693</td>
</tr>
</tbody>
</table>

Table 2. Comparisons of the transverse normal and shear stresses at mid-plane for \( n = 0.1, 0.5, 1, 2, \) and 10.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \sigma_{xx} ) (Pa) (at point ( B ))</th>
<th>( \sigma_{yy} ) (Pa) (at point ( C ))</th>
<th>( \sigma_{zz} ) (Pa) (at point ( A ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical</td>
<td>RZE (^{2,2})</td>
<td>Analytical</td>
<td>RZE (^{2,2})</td>
</tr>
<tr>
<td>0.1</td>
<td>1.6156</td>
<td>1.5957</td>
<td>2.1033</td>
</tr>
<tr>
<td>0.5</td>
<td>1.6147</td>
<td>1.5966</td>
<td>2.1042</td>
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<td>1</td>
<td>1.6309</td>
<td>1.6163</td>
<td>2.1309</td>
</tr>
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<td>1.6842</td>
<td>1.6708</td>
<td>2.2036</td>
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<td>10</td>
<td>1.7866</td>
<td>1.7683</td>
<td>2.3344</td>
</tr>
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Also, the present approach captures contraction in the thickness direction as shown in Fig. 8(c). As seen, the through thickness transverse displacement variation at the plate center coincides with the analytical solution.

Figure 9 illustrates the comparison of through-thickness in-plane stress components, \( \sigma_{xx} \) and \( \sigma_{yy} \), calculated at the selected points (Fig. 7) for the compositional gradient component of \( n = 1 \). As seen, the results indicate very close agreements between the analytical solution and RZE\(^{2,2}\). Also, the in-plane stress levels reduce towards the bottom surface which is successfully captured by the RZE\(^{2,2}\). The variations of transverse shear and normal stress components at the evaluation points (Fig. 7) for the compositional gradient component of \( n = 1 \) are presented in Fig. 10. It is apparent that the transverse stress variation, \( \sigma_{zz} \), at the center of the plate indicates remarkable agreement with the analytic solution (Fig. 10(c)). Also, it is evident that RZE\(^{2,2}\) successfully captures the quadratic form of the transverse shear stresses through the thickness of the FG plate. The middle of the plate experiences higher transverse shear stresses, and the effect of the transverse shear stresses decreases uniformly towards the top and bottom surfaces of the FG plate.
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Figure 8. Comparisons of the in-plane and transverse displacements, (a) $u$, (b) $v$ and (c) $w$ for $n=1$.

Figure 9. The through-thickness variations of in-plane stress components (a) $\sigma_{xx}$ at point A, and (b) $\sigma_{yy}$ at point C for $n=1$.

Figure 10. The through-thickness variations of transverse shear stress components (a) $\sigma_{yz}$ at point B, (b) $\sigma_{zy}$ at point C, and transverse normal stress component (c) $\sigma_{zz}$ at point A for $n=1$. 

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Figure 11 shows the in-plane and transverse displacement variations at the selected points (Fig. 7) with compositional gradient exponents of \( n = 0.1, 0.5, 1, 2, \) and 10. As seen, the compositional gradient exponent has an apparent effect on both in-plane and transverse displacement levels; however, it plays a minor role on the displacement profiles. As SiC (ceramic) constituent is increased in the material composition, the FG plate exhibits a stiffer behavior. Thus, the in-plane and transverse displacement components experience higher levels with changing the compositional gradient component from the metal-rich (\( n = 0.1 \)) to ceramic-rich (\( n = 10 \)).

Figure 12 depicts the transverse normal and transverse shear stress variations through the thickness of the plate at the selected points (Fig. 7) with compositional gradient exponents of \( n = 0.1, 0.5, 1, 2, \) and 10. A uniform stress variation is observed by functionally tailoring the material properties through the thickness of the plate. The transverse shear stresses at the top and bottom surfaces vanish, and they become maximal in the middle of the FG plate. The through-thickness material variation has negligible effects on the transverse shear stress profiles. Moreover, the maximum transverse shear stress values tend to increase in the ceramic-rich (\( n = 10 \)) material composition comparing to the balanced (\( n = 1 \)) and metal-rich (\( n = 0.1 \)) material compositions. The differences among the transverse normal stress components are negligible for each compositional gradient exponent (Fig. 12(c)). The transverse normal stresses are evident at the top surface where the load is applied whilst the bottom surface of the FG plate experiences zero transverse normal stresses.

Figure 11. The through-thickness variations of the in-plane and transverse displacements, (a) \( u \), (b) \( v \) and (c) \( w \).

Figure 12. The through-thickness variations of transverse shear stress components (a) \( \sigma_{xz} \) evaluated at point \( B \), (b) \( \sigma_{yz} \) evaluated at point \( C \), and transverse normal stress component (c) \( \sigma_{zz} \) evaluated at point \( A \).
9. CONCLUSIONS

In this study, the flexural analysis of the FG plates using \{2,2\}-refined zigzag plate theory was investigated. Most plate elements available in the literature neglect the transverse stretching through the thickness. This element adopts the quadratic through-thickness variation of the transverse displacement components. The anisoparametric shape functions were employed to eliminate the geometric locking. Since the element assumes the constant shear strain variation within each layer, it removes the requirement for the shear correction factor. The governing equations of the FG plate and the boundary conditions were derived by employing the principle of virtual work. The validity of the RZE\(^{2,2}\) was presented for the simply supported FG plate under transverse bi-sinusoidal load. The present element accurately captured the displacement and stress variations through the thickness of the FG plate having a balanced material composition. The through-thickness mechanical properties of the FG plate were evaluated by using the Mori-Tanaka homogenization scheme [26]. The influence of the material variation of the FG plate on the displacement and stress variations were elucidated. It was observed that the through-thickness material variation was influential on the peak displacement and stress levels whereas it had a minor effect on both displacement and stress profiles. Increasing the metal phase in the material composition resulted in both in-plane and transverse displacements components to increase in comparison with those in the FG plates having balanced and ceramic-rich material compositions. Also, increases were observed in the peak stress levels with enriching the material composition with the ceramic phase.

REFERENCES


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